# First and Second Order Linear Ordinary Differential Equations with Constant Coefficients

*This is revision material. Its purpose is to remind you of various topics relevant to this course, while emphasising the language and terminology associated with differential equations*

## *1 Differential Equations as models for the Dynamics of Physical Systems*

### 1.1 Mechanical systems

The dynamics of mechanical systems are determined by Newton's second law

$$m\ddot{x} = F\ . \tag{1.1}$$

Here $m$ is the mass of the object, $x$ is its displacement from some origin, and $F$ is the force experienced by the body.

Friction is modelled by a force proportional to, and in opposition to, the velocity $\dot{x}$:

$$F_{\mathrm{f}} = -\lambda\dot{x}\ . \tag{1.2}$$

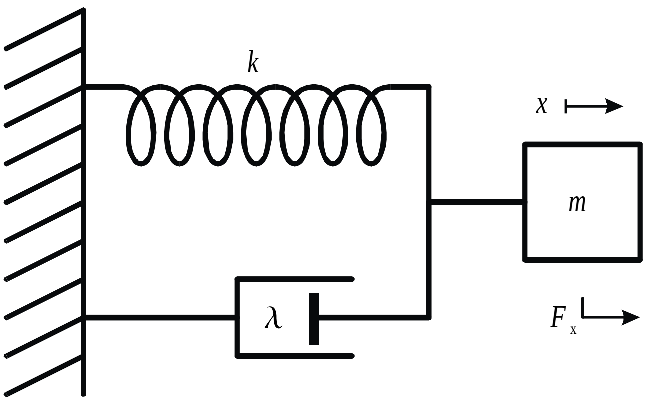
Here $\lambda$ is the friction constant. (It is usually stated that taking the friction force *proportional* to the velocity is simply an approximation to reality. A deeper discussion, together with a justification of the linear assumption is contained in BPC's book Classical Mechanics.)

A spring, obeying Hook's law, provides a restoring force proportional to, and in opposition to, the displacement from an equilibrium position. Taking the equilibrium position at the origin then allows us to write

$$F_{\mathrm{s}} = -kx \tag{1.3}$$

and $k$ is known as the spring constant.

The mass might also be subject to an externally impressed force $F_{\mathrm{x}}$, which may well depend on time.




*Mechanical model of a damped oscillator*

Combining these forces into the Newton law expression gives the equation of motion

$$m\ddot{x} + \lambda\dot{x} + kx = F_{\mathrm{x}}\ . \tag{1.4}$$

By convention the dependent coordinate *x* together with its derivatives is placed on the left and the external force is placed on the right hand side of the equation: *cause* on the right, *effect* on the left.

We observe that our equation of motion is a

- *linear*
- *inhomogeneous*
- *second order*
- *ordinary*
- *differential equation,* with
- *constant coefficients*.

If there is no external force, $F_{\mathrm{x}} = 0$ then the equation of motion becomes

$$m\ddot{x} + \lambda\dot{x} + kx = 0\,. \tag{1.5}$$

This is now a *homogeneous* equation, a

- *linear*
- *homogeneous*
- *second order*
- *ordinary*
- *differential equation, with*
- *constant coefficients*.

We note that the homogeneous equation describes a *free* or *isolated* system. It is the *source* $F_{\mathrm{x}}$ of the inhomogeneous equation that accounts for the interaction with the external force.

*1.1.1 Special cases*

1 No spring; just the friction; no external force

Here the equation of motion is

$$m\ddot{x} + \lambda\dot{x} = 0\,,$$

which can be rewritten in terms of the velocity as

$$m\dot{v} + \lambda v = 0\,.$$

This is a *first order* equation in the velocity.

2 No friction; just the spring force; no external force

Here the equation of motion is

$$m\ddot{x} + kx = 0\,.$$

This is the equation for the (undamped) simple harmonic oscillator.

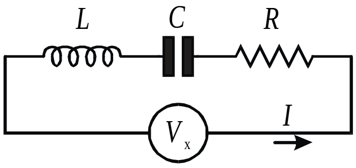
## 1.2 Electrical systems

We consider, specifically, a series LCR circuit. The voltage – current relations for the inductor, resistor and capacitor are:

$$\begin{array}{ll} \text{inductor} & V = L\dfrac{\mathrm{d}I}{\mathrm{d}t} \\ \text{resistor} & V = RI \\ \text{capacitor} & V = \dfrac{1}{C}Q = \dfrac{1}{C}\displaystyle\int I\mathrm{d}t\,. \end{array} \tag{1.6}$$

Including an externally applied voltage $V_x$, which might depend on time, the voltages are added to give the 'equation of motion'

$$L\dot{I} + RI + \frac{1}{C}\int I\mathrm{d}t = V_x \,. \tag{1.7}$$




*Driven series LCR circuit*

The equation is brought into "standard form" by differentiating once, giving:

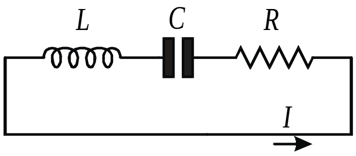
$$L\ddot{I} + R\dot{I} + \frac{1}{C}I = \dot{V}_x \,. \tag{1.8}$$

As in the mechanical case we have a

- *linear*
- *inhomogeneous*
- *second order*
- *ordinary*
- *differential equation,* with
- *constant coefficients*.

If there is no external voltage (or a constant voltage), $\dot{V}_x = 0$ then the equation of motion becomes

$$L\ddot{I} + R\dot{I} + \frac{1}{C}I = 0 \,.$$




*Free LCR circuit*

This is now a *homogeneous* equation, a

- *linear*
- *homogeneous*
- *second order*
- *ordinary*
- *differential equation,* with
- *constant coefficients*.

We note that the homogeneous equation describes a *free* or *isolated* system. It is the *source* $V_x$ of the inhomogeneous equation which accounts for the interaction with the external voltage.

### *1.2.1 Special cases*
The LR and RC circuits are described by first order equations and the LC circuit is described by the second order SHO equation with no damping (friction).

# ***2 Solutions***

## 2.1 First order equations
The "standard form" for the first order ordinary differential equation with constant coefficients is

$$\dot{x} + \gamma x = f(t). \tag{2.1}$$

### *2.1.1 Homogeneous equation*
The homogeneous equation is

$$\dot{x} + \gamma x = 0 \tag{2.2}$$

and this is described in terms of *one* parameter $\gamma$. Note that the dimensions of the parameter $\gamma$ is 1/time.

The solution to this equation is

$$x = Ae^{-\gamma t} \tag{2.3}$$

where $A$ is a constant. Since this is a first order equation there is just one adjustable constant; this may be determined from the *initial* conditions: the state at $t = 0$.

Observe that the solution satisfies *time translation invariance*: wherever the origin of time is taken, the solution has the same mathematical form

### *2.1.2 Inhomogeneous equation*
The inhomogeneous equation

$$\dot{x} + \gamma x = f(t) \tag{2.4}$$

has a solution which is the sum of:

- the complementary function CF and
- the particular integral PI.

The complementary function is the solution of the corresponding homogeneous equation. The fact that the CF can be added to any solution follows from the linearity of the equation.

The particular integral is a solution that depends explicitly on the source $f(t)$.

We have seen above that the complementary function is

$$\mathrm{CF} = Ae^{-\gamma t}$$

The particular integral may be expressed as

$$\begin{aligned} \mathrm{PI} &= e^{-\gamma t}\int^t e^{\gamma\tau} f(\tau)\mathrm{d}\tau \\ &= \int^t e^{\gamma(\tau - t)} f(\tau)\mathrm{d}\tau. \end{aligned} \tag{2.5}$$

So the general solution to the inhomogeneous first order equation is

$$x(t) = e^{-\gamma t}\left[A + \int^t e^{\gamma\tau} f(\tau)\mathrm{d}\tau\right]. \tag{2.6}$$

### *2.1.3 Physical interpretation of the Complementary Function and the Particular Integral*

The *Complementary Function* is:

- independent of the source and
- depends on the initial conditions.

Thus it contains the *transient behaviour* which follows from the way the system was "started off". On physical grounds this will die away after sufficient time.

The *Particular Integral* is

- independent of the initial conditions, and
- it depends on the source.

This is the form of the solution at times sufficiently long that the initial transients have died out. When the source is some sort of continuous function the particular integral may be referred to as a *steady state* solution.

⌈Questions for thought:
a Why Differential Equations?
Derivatives are that natural way to describe changing states – dynamics, when the change occurs "smoothly".

b Why Second order equations?
The answer is probably in two parts. First order equations can only describe growth and decay; they cannot describe "steady state", repetitive behaviour (time translation invariance) such as sines and cosines. But why don't we need higher order equations. This might be understood by appealing to Occam's razor. Newton's law is fundamentally second order. ⌋

## 2.2 Second order equations

The "standard form" for the second order ordinary differential equation with constant coefficients is

$$\ddot{x} + \lambda\dot{x} + \omega_0^2 x = f(t). \tag{2.7}$$

### *2.2.1 Homogeneous equation*

The homogeneous equation is

$$\ddot{x} + \lambda\dot{x} + \omega_0^2 x = 0 \tag{2.8}$$

and this is described in terms of *two* parameters $\lambda$ and $\omega_0$. The reason for the form of the coefficient of *x* will become apparent. Note that the dimensions of both parameters $\lambda$ and $\omega_0$ must be 1/time. (mathematically convenient)

A special case of this equation occurs when there is no friction term $\lambda$. Then we have the SHO equation

$$\ddot{x} + \omega_0^2 x = 0$$

and this has solution

$$x = A\cos\omega_0 t + B\sin\omega_0 t\ , \tag{2.9}$$

or equivalently

$$x = C\cos\left(\omega_0 t + \varphi\right). \tag{2.10}$$

Alternatively we might express the solution in complex form

$$x = De^{\mathrm{i}\omega_0 t} + Ee^{-\mathrm{i}\omega_0 t}. \tag{2.11}$$

This is a second order equation so there are two adjustable constants; these may be determined from the *initial* conditions, $x(0)$ and $\dot{x}(0)$ or from the value of *x* at two different times.

The solutions above indicate the significance of $\omega_0$ as the angular frequency of the (friction-free) oscillations.

*2.2.2 General second order homogeneous equation*

We now consider the equation in standard form

$$\ddot{x} + \lambda\dot{x} + \omega_0^2 x = 0.$$

(Clearly the solution must reduce to the above friction-free case in the limit $\lambda \to 0$.)

The general solution is found through the substitution $x \propto e^{pt}$. Then the derivatives are found as

$$\dot{x} = px, \qquad \ddot{x} = p^2 x.$$

These are substituted into the (differential) equation of motion to give the algebraic equation

$$p^2 + \lambda p + \omega_0^2 = 0$$

whose solutions are

$$p_\pm = \frac{-\lambda \pm \sqrt{\lambda^2 - 4\omega_0^2}}{2}. \tag{2.12}$$

We have two solutions corresponding to the two signs of the square root in the expression for *p* and the general solution is

$$x = Ae^{p_+ t} + Be^{p_- t}.$$

*2.2.3 Light damping*

In this case $\lambda$ is small; $\lambda << \omega_0$ so that the square root is imaginary. It is then better to express *p* as

$$\begin{aligned} p &= -\frac{\lambda}{2} \pm \frac{\mathrm{i}}{2}\sqrt{4\omega_0^2 - \lambda^2} \\ &= -\frac{\lambda}{2} \pm \mathrm{i}\sqrt{\omega_0^2 - (\lambda/2)^2}. \end{aligned} \tag{2.13}$$

The solutions are of the form

$$e^{-t/\tau}e^{\mathrm{i}\omega t}, \qquad e^{-t/\tau}e^{-\mathrm{i}\omega t}, \tag{2.14}$$

or the linear combinations

$$e^{-t/\tau}\cos\omega t, \qquad e^{-t/\tau}\sin\omega t \tag{2.15}$$

where

$$\tau = 2/\lambda, \qquad \omega = \sqrt{\omega_0^2 - (\lambda/2)^2} \tag{2.16}$$

and the general solution is

$$x(t) = e^{-t/\tau}\left\{A\cos\omega t + B\sin\omega t\right\}. \tag{2.17}$$

We see that $\lambda$ (the coefficient of $\dot{x}$) determines the time constant $\tau$ of the decay. The oscillation angular frequency is determined mainly by $\omega_0$ (the coefficient of $x$), but there is a small reduction in this frequency caused by the damping. Often this may be neglected.

*2.2.4 Q factor (quality factor)*

The $Q$ factor is a dimensionless quantity, which may be used to quantify the dissipation (damping) of an oscillator. This is called the *quality factor*, or sometimes the *selectivity*. High $Q$ means low damping and *vice-versa*.

The dimensions of the damping parameter $\lambda$ are the same (1/time) as those of the angular frequency $\omega_0$. Thus the ratio $\omega_0/\lambda$ is dimensionless. We define the $Q$ factor by

$$Q = \frac{\omega_0}{\lambda}. \tag{2.18}$$

We see that $Q$ is high for small damping.

Incorporating the $Q$ factor, the damped oscillator differential equation may be parameterised in terms of the two quantities $\omega_0$ and $Q$. Then the damping time constant and the oscillation (angular) frequency may be expressed as

$$\tau = \frac{2Q}{\omega_0}, \qquad \omega = \omega_0\sqrt{1-\left(\frac{1}{2Q}\right)^2}\ . \tag{2.19}$$

*2.2.5 Properties of* Q

We list two important properties of the $Q$ factor. After considering forced oscillations we shall be in a position to list some more.

- The number of oscillation cycles for the amplitude to decrease to $1/e$ of its value is $Q/\pi$.
- $Q = 2\pi\times$ stored energy / energy lost per cycle.

It is often convenient to parameterise an oscillator in terms of its resonant frequency and its $Q$ factor. This casts the description in abstract/general terms.

*2.2.6 Frequency shift*

The expression for the frequency shift is

$$\omega = \omega_0\sqrt{1-\left(\frac{1}{2Q}\right)^2}\ .$$

For large-ish $Q$ we can expand the oscillation frequency in inverse powers of $Q$. In this way we have

$$\omega = \omega_0\left(1-\frac{1}{8Q^2}+\ldots\right). \tag{2.20}$$

So even for a modest $Q$ factor of $\sim 10$, the shift in the oscillation frequency is very small. Thus in many practical applications $\omega_0$ is taken to be the angular frequency, with negligible error.

### *1.2.7 Heavy damping*

When the damping is large, then $\lambda >> \omega_0$. In that case the two values of $p$ are real and unequal:

$$p = \frac{-\lambda \pm \sqrt{\lambda^2 - 4\omega_0^2}}{2}.$$

This means that there is no oscillation. One value of $p$ will be close to $\lambda$ while the other will be very small. These values are best expressed in terms of $Q$, which is now very small. Since

$$p = \frac{-\lambda \pm \lambda\sqrt{1-4Q^2}}{2}, \tag{2.21}$$

by expanding in powers of $Q$ we obtain

$$\begin{aligned} p_1 &= -\lambda\left(1 - Q^2 + \ldots\right) \\ p_2 &= -Q^2\lambda + \ldots \;. \end{aligned} \tag{2.22}$$

In the limit of very small $Q$ we see that $p_1$ will tend to $\lambda$ while $p_2$ will tend to zero. The general solution

$$x(t) = Ae^{p_1 t} + Be^{p_2 t} \tag{2.23}$$

will tend to the limit of the solution to the first order equation

$$\dot{x} + \lambda x = 0,$$

namely

$$x(t) = Ae^{-\lambda t}.$$

This happens by the coefficient $B$ going to zero.

## 2.3 Forced oscillations

We now return to the standard form second order ordinary differential equation with constant coefficients in inhomogeneous form: that is, with a source term $f(t)$

$$\ddot{x} + \lambda\dot{x} + \omega_0^2 x = f(t).$$

We know that the general solution is the sum of the complementary function and a particular integral. And we know the complementary function may be written

$$e^{-t/\tau}\left\{A\cos\omega t + B\sin\omega t\right\}.$$

The new problem is to find the particular integral, a function of the driving force $f(t)$. For long times ($t \gg \tau$) the transient behaviour of the CF will have decayed away and the remaining behaviour will be solely that of the PI.

### *2.3.1 Sinusoidal driving force*

For a general driving force the motion $x(t)$ will be complicated. We will not, here, discuss the general procedures for finding the PI for a general source term. Instead we will examine the solution corresponding to a sinusoidal driving force. We take

$$f(t) = f_0\cos\omega t. \tag{2.24}$$

It is important to note that here $\omega$ is a general angular frequency; it is not the free oscillation frequency of above.

We are looking for a steady solution to the equation

$$\ddot{x} + \lambda\dot{x} + \omega_0^2 x = f_0 \cos\omega t\,. \qquad (2.25)$$

The *linearity* of this equation means that the solution $x(t)$ will be a steady oscillation at the driving angular frequency $\omega$. And the amplitude will be proportional to the magnitude of the force $f_0$.

There is a choice of ways to represent this solution. The two most important representations are the amplitude/phase and the in-phase/quadrature descriptions.

*2.3.2 In-phase/quadrature description*

In this case the solution is written as

$$x(t) = f_0\left\{A(\omega)\cos\omega t + B(\omega)\sin\omega t\right\}. \qquad (2.26)$$

The cos and sin give the time dependence. The ‘constants’ *A* and *B* will depend on the driving frequency.

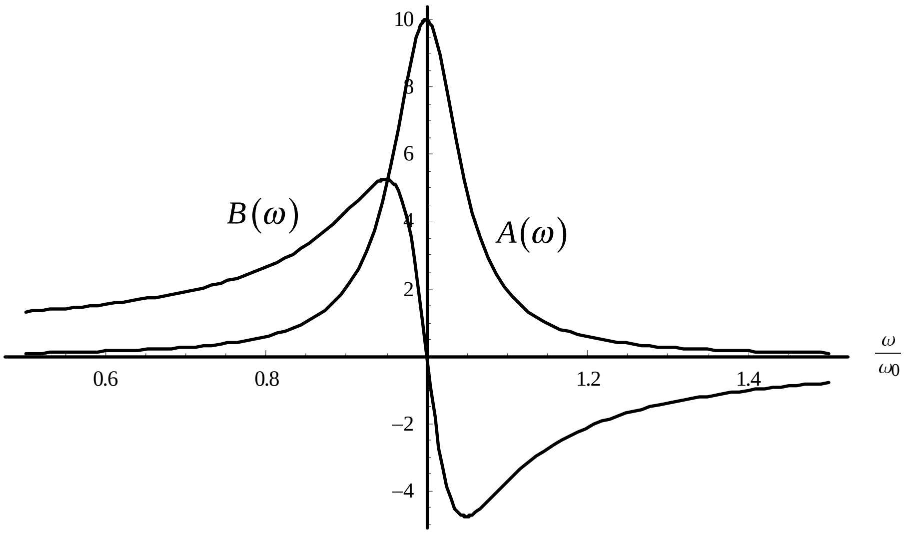
By substituting this general solution into the original equation and solving for *A* and *B*, one finds

$$\frac{x(t)}{f_0} = \frac{\omega_0^2-\omega^2}{\left(\omega_0^2-\omega^2\right)^2+(\lambda\omega)^2}\cos\omega t + \frac{\lambda\omega}{\left(\omega_0^2-\omega^2\right)^2+(\lambda\omega)^2}\sin\omega t\,. \qquad (2.27)$$

The coefficient of $\cos\omega t$, that is $A(\omega)$, gives the part of the response that is *in-phase* with the excitation $f_0\cos\omega t$. The coefficient of $\sin\omega t$, that is $B(\omega)$, gives the response that is in *quadrature* with the excitation: the part that is 90° out of phase with the excitation. Thus we write

$$\begin{aligned} A(\omega) &= \frac{\left(\omega_0^2-\omega^2\right)}{\left(\omega_0^2-\omega^2\right)^2+(\lambda\omega)^2} \\ B(\omega) &= \frac{\lambda\omega}{\left(\omega_0^2-\omega^2\right)^2+(\lambda\omega)^2}. \end{aligned} \qquad (2.28)$$

These




*in-phase and quadrature response of oscillator*

*2.3.3 Amplitude/phase description*

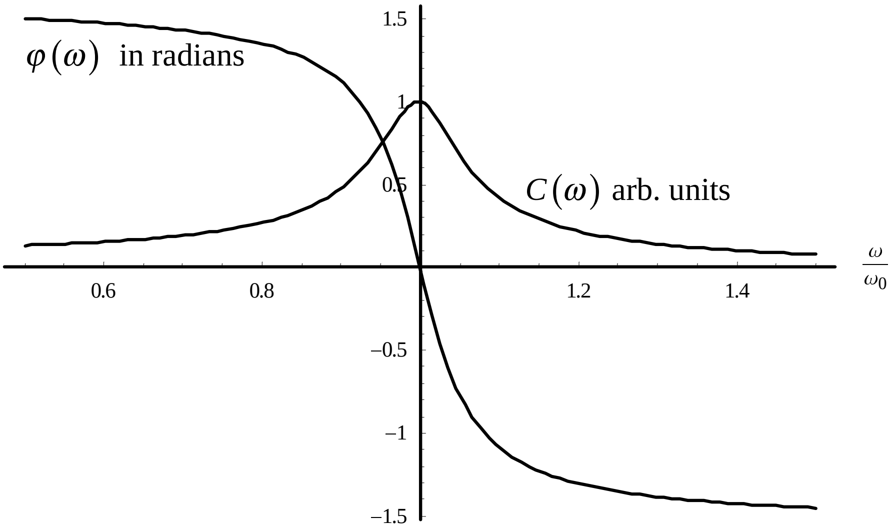
In this case the solution is written as

$$x(t) = f_0 C(\omega)\cos\left(\omega t + \varphi(\omega)\right) \tag{2.29}$$

and here we must determine the functions $C(\omega)$ and $\varphi(\omega)$. This is most simply done in terms of the response functions $A(\omega)$ and $B(\omega)$ already found. We have

$$\begin{aligned} C(\omega) &= \left\{A(\omega)^2 + B(\omega)^2\right\}^{1/2} \\ \varphi(\omega) &= \tan^{-1}\left(B(\omega)/A(\omega)\right) \end{aligned} \tag{2.30}$$

and the




*amplitude / phase description of oscillator*

The phase $\varphi$ is zero at resonance; this is often used as a criterion for resonance. As one sweeps through resonance the phase moves from $\pi/2$ through zero to $-\pi/2$. One should also observe that the width of $C(\omega)$ is greater than the width of $A(\omega)$ in the vicinity of resonance. Clearly the height of $C(\omega)$ is the same as that of $A(\omega)$ at resonance.

*2.3.4 More properties of* Q

In the frequency domain $Q$ measures the sharpness of the resonance. The $Q$ factor is the reciprocal fractional width of the resonance

$$Q = \frac{\omega_0}{\Delta\omega}, \tag{2.31}$$

where the resonance width $\Delta\omega$ is (for a sharp resonance) the frequency difference between the points that

- the phase $\varphi(\omega)$ is ±45° of its value (zero) on resonance, or
- the amplitude response $C(\omega)$ drops to $1/\sqrt{2}$ of its value on resonance, or
- the in-phase response $A(\omega)$ drops to half its value on resonance.

### *2.3.5 Complex representation*

In making contact with the real world one expects to be working with *real* coordinates $x(t)$ and forces $f(t)$. The *linearity* of the equations we have been considering allows us to exploit the mathematical convenience of complex variables. At the simplest level it is clear that if a complex $x(t)$ arises as the mathematical solution from a complex $f(t)$, then the response to a force which is the real part of this complex $f(t)$ will be the real part of the corresponding complex $x(t)$.

There is, however, a very useful generalisation of this idea that relies on the deMoivre relation

$$\cos\varphi + \mathrm{i}\sin\varphi = e^{\mathrm{i}\varphi}.$$

The response to the force

$$f(t) = f_0\cos\omega t \tag{2.32}$$

is the displacement

$$x(t) = f_0 C(\omega)\cos(\omega t + \varphi). \tag{2.33}$$

Similarly, the response to a force

$$f(t) = f_0\sin\omega t \tag{2.34}$$

will be the displacement

$$x(t) = f_0 C(\omega)\sin(\omega t + \varphi). \tag{2.35}$$

Thus applying the linearity rule above implies that the (mathematical) response to the complex force

$$\begin{aligned} f(t) &= f_0\cos\omega t + \mathrm{i}f_0\sin\omega t \\ &= f_0 e^{\mathrm{i}\omega t} \end{aligned} \tag{2.36}$$

is the complex displacement

$$\begin{aligned} x(t) &= f_0 C(\omega)\cos(\omega t + \varphi) + \mathrm{i}f_0 C(\omega)\cos(\omega t + \varphi) \\ &= f_0 C(\omega) e^{\mathrm{i}(\omega t + \varphi)} \\ &= f_0 C(\omega) e^{\mathrm{i}\varphi} e^{\mathrm{i}\omega t}. \end{aligned} \tag{2.37}$$

This expression may be interpreted as saying that the response to the force

$$f(t) = f_0 e^{\mathrm{i}\omega t} \tag{2.38}$$

is the displacement

$$x(t) = f_0\tilde{C}(\omega) e^{\mathrm{i}\omega t}. \tag{2.39}$$

Here we are using a *complex* response function $\tilde{C}(\omega)$

where

$$\begin{aligned} \tilde{C}(\omega) &= C(\omega) e^{\mathrm{i}\varphi} \\ &= A(\omega) + \mathrm{i}B(\omega). \end{aligned} \tag{2.40}$$

We see that both amplitude and phase information are contained in the complex response function. And of course the mathematics of complex exponentials is so much simpler than that of sines and cosines continually converting into each other.